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MODULAR GROUP RINGS OF THE FINITARY SYMMETRIC GROUP

BY

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ABSTRACT

The paper discusses approaches to constructing two-sided ideals of the modular group algebra of finitary symmetric group.

The aim of these notes is to collect information about the group rings of the finitary symmetric group over fields of prime characteristic. We denote by S_n the symmetric group on symbols $\{1, \ldots, n\}$ and by S the finitary symmetric group on an infinite set of symbols. Recall that the finitary symmetric group is the group of all finitary permutations of an infinite set. A permutation of a set is called finitary if it fixes all but finitely many elements of the set.

The finitary symmetric group plays an important role in the theory of identities of algebras, see [1, 11, 12, 15]. In particular, in the theory of trace identities as developed in Razmyslov [11, 12] one needs to know the two-sided ideals of the group ring of the finitary symmetric group. For a ground field of characteristic 0 there exists a complete description of the two-sided ideals, see [6, 11]. However, very little is known about ideals in the group ring of the finitary symmetric group over fields of prime characteristic. In particular, even the very general question whether two-sided ideals satisfy the ascending chain condition remains open.

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I would like to attract the attention of a wider circle of mathematicians to this topic.

SOME NOTATION. RG denotes the group ring of a group G over a ring R. All modules considered are left modules. 1_G stands for the trivial one-dimensional representation of G or for the trivial module of dimension 1. If H is a group and $G \subset H$ then 1_G^H is the induced representation or the induced module. More generally, if ϕ is a representation of G (RG-module) then ϕ^H is the induced representation (module). For an RG-module L, we write $\mathrm{Ann}_{RG}(L)$ for the annihilator of L in RG. A_n denotes the alternating group on n symbols. S_n and S have been introduced above.

1. Annihilators of induced modules

The most natural way of constructing ideals in group rings is to take annihilators of RG-modules in RG. For infinite groups most modules have zero annihilators, and the problem of finding modules with nonzero annihilators is not easy. For the finitary symmetric group there exist many modules with nontrivial annihilators. If the ground field P is of characteristic 0, all two-sided ideals of PG are known. If P is of prime characteristic, the problem of determining ideals seems to be very difficult. Possibly, it would be helpful to study first the annihilators of induced modules. Essential information may be obtained from [16] and [7]. In [16] the subgroups X of S, such that the annihilator of the PS-module 1_X^S is not zero, are determined. In [7] it is proved that this annihilator is a prime ideal for certain X. The latter will be discussed in the next section.

THEOREM 1 ([16]): *Let P be an arbitrary field and Ω an infinite set. Let $X \subset S = S(\Omega)$ be a subgroup. Then the following conditions are equivalent:*

(i) *the annihilator of the induced PS-module 1_X^S is not zero;*

(ii) *X contains a normal subgroup of finite index which is the product of finitely many finitary alternating groups $\mathrm{Alt}(\Omega_1), \dots, \mathrm{Alt}(\Omega_n)$ with disjoint infinite Ω_i's ($i = 1, \dots, n$) whose complement $\Omega - (\Omega_1 \cup \cdots \cup \Omega_n)$ is finite (or empty).*

The way to prove this Theorem is based on the following

PROPOSITION 1: *Let G be an arbitrary group and X a subgroup. Let L be a PX-module and L^G the induced PG-module. Suppose that $\mathrm{Ann}(L^G) \neq \{0\}$. Then there exists a finite subset F of G such that $X \cap g^{-1}Fg \neq \{1\}$ for all $g \in G$.*

Proof: Let $0 \neq \mathrm{Ann}(L^G)$. Write $a = \sum_{i=1}^{k} a_i g_i \in \mathrm{PG}$ with $g_i \in G$, $0 \neq a_i \in P$. Take for F the set $g_i^{-1} g_j$ $(i, j \in \{1, \dots, k\})$. As L^G is a direct sum of $g \otimes L$ where g runs over a transversal T of the left cosets G/X, $aL^G = \{0\}$ implies $ag \otimes L = \{0\}$ for all $g \in G$. Hence for any $l \in L$ we have $\sum_{i=1}^{k} a_i g_i g \otimes l = 0 = \sum_{i=1}^{k} a_i g^{-1} g_i g \otimes l$. If we write $g^{-1} g_i g = t_i x_i$ with $t_i \in T$, $x_i \in X$ then $0 = \sum_{i=1}^{k} a_i t_i \otimes x_i l$. It follows that $t_i = t_j$ at least for one pair $i, j \in \{1, \dots, k\}$ with $i \neq j$. So $g^{-1} g_i g x_i^{-1} = g^{-1} g_j g x_j^{-1}$ whence $1 \neq g^{-1} g_j^{-1} g_i g = x_j^{-1} x_i \in X$. Hence we see that $g^{-1} F g \cap X \neq \{1\}$, as desired. ∎

It turned out [16] that for subgroups X of S the condition (ii) of Theorem 1 is equivalent to the existence of a finite set F of G such that $X \cap g^{-1} F g \neq \{1\}$ for all $g \in G$. So, if $\mathrm{Ann}(L^G) \neq \{0\}$, where L is a PX-module, then X is as in (ii) of Theorem 1. However, for these X it remains unclear for which PX-modules L one has $\mathrm{Ann}(L^G) \neq \{0\}$. It is clear from the above that a necessary condition is $\mathrm{Ann}_{PX}(L) \neq \{0\}$).

2. A PS-module constructed via the free associative algebra

Let $P(X)$ denote the free associative algebra over a field P with the set $X = \{x_1, x_2, \dots\}$ of free generators. We call elements of $P(X)$ polynomials in the indeterminates $x_1, x_2, \dots$, which, however, do not commute with each other. Let V be the subspace of linear polynomials in $x_1, x_2, \dots$. By $P_k(X)$ we denote the subalgebra of $P(X)$ generated by $x_1, \dots, x_k$ and $V_k \subset V$ the subspace of linear polynomials in $x_1, \dots, x_k$. The action of the general linear group $\mathrm{GL}_k(P)$ on V_k extends to $P_k(X)$ in the natural way. Denote by V_{kn} the space of polynomials of $P_k(X)$ of degree n. The monomials $x_{i_1} x_{i_2} \dots x_{i_n}$ $(i_1, \dots, i_n \in \{1, \dots, k\})$ form a basis of V_{kn}. There are two natural actions of symmetric groups on V_{kn}. The group S_k acts on the monomials $x_{i_1} x_{i_2} \dots x_{i_n}$ by permuting indeterminates, and S_n acts on them by permuting positions which are occupied by x's. We now fix our attention on the second case.

Let $B(j_1, \dots, j_k)$ with $j_1 + \cdots + j_k = n$ be the set of all monomials of degree j_i in x_i $(i = 1, \dots, k)$ and $M(j_1, \dots, j_k)$ the P-span of $B(j_i, \dots, j_k)$. Observe that we allow for convenience that some j_i may be zero. It is clear that $M(j_1, \dots, j_k)$ is a permutation PS_n-module and S_n is transitive on $B(j_1, \dots, j_k)$. The stabilizer of any monomial in S_n is a Young subgroup $Y(j_1, \dots, j_k)$. It follows that $M(j_1, \dots, j_k)$ is an induced module $1_{Y(j_1, \dots, j_k)}$. Furthermore, V_{kn} is a permutation PS_n-module which is a direct sum of various $M(j_1, \dots, j_k)$

with $j_1 + \cdots + j_k = n$. Observe that the S_n-orbit of $B(j_1, \ldots, j_k)$ contains that monomial f, which has the indeterminate x_i in the positions $j_1 + \cdots + j_{i-1} + l$ where $0 < l \leq j_i$ (it may be written as $x_1^{j_1} \ldots x_k^{j_k}$). If $(j_1', \ldots, j_k')$ is a permutation of $(j_1, \ldots, j_k)$, then $M(j_1', \ldots, j_k')$ is isomorphic to $M(j_1, \ldots, j_k)$.

If S_n is naturally embedded into S_{n+1}, then $V_{k(n+1)} \mid S_n$ is a direct sum of k PS_n-modules each isomorphic to V_{kn}. Let $I(n,k)$ denote the annihilator of V_{kn} in PS_n. Then $I(n,k) \subset I(n+1,k)$. Hence $\bigcup_n I(n,k) = I(k)$ is a two-sided ideal of PS. Let $V(k)$ denote the direct limit of V_{kn}. Then $I(k) = \mathrm{Ann}_{PS} V(k)$.

THEOREM 2 (Formanek and Procesi [7, §3])): *$PS/I(k)$ contains no nilideals. In particular, $I(k)$ is a semiprime ideal of PS.*

We show that $I(k)$ is the annihilator of an induced module.

PROPOSITION 2: *Let A_k be the annihilator of the induced module 1_X^S, where X is the Young subgroup associated with the partition of Ω in k infinite subsets. Then $I(k) = A_k$.*

Proof: Put $A(k,n) = A_k \cap PS_n$. We show that $A(k,n) = I(k,n)$. Recall that the restriction $1_X^S \mid S_n$ is a direct sum of modules isomorphic to $(1_{S_n \cap sXs^{-1}})^{S_n}$ where s runs over a transversal of double cosets $S_n \setminus S/X$. Let $\Omega_n = \mathrm{supp}(S_n)$ and let $\Delta_1, \ldots, \Delta_k$ be the orbits of X on Ω. Put $j_i(s) = |s\Delta_i \cap \Omega_n|$, $i = 1, \ldots, k$. Then $S_n \cap sXs^{-1}$ is the Young subgroup $Y(j_1(s), \ldots, j_k(s))$ with $j_1 + \cdots + j_k = n$. It is clear that every Young subgroup $Y(j_1, \ldots, j_k)$ with $j_1 + \cdots + j_k = n$ is of the form $S_n \cap sXs^{-1}$ for a suitable $s \in S$. Hence $1_X^S \mid S_n$ and V_k both are direct sums of modules isomorphic to $M(j_1, \ldots, j_k)$ with $j_1 + \cdots + j_k = n$. Hence $\mathrm{Ann}_{PS_n} 1_X^S \mid S_n = \mathrm{Ann}_{PS_n} V_k = I(n,k)$. Hence $\mathrm{Ann}_{PS} 1_X^S = \mathrm{Ann}_{PS} V_k = I(k)$. ∎

It would be good to know more about ideals $\mathrm{Ann}_{PS} 1_X^S$ as well as about the relationship of these ideals to other ideals. For example, what are the X with $\mathrm{Ann}_{PS} 1_X^S$ semiprime? Is the square of this ideal different from it?

We know no way to solve the following

PROBLEM: *Is it true that two-sided ideals of PS satisfy the ascending chain condition?*

For P of characteristic 0 this is well known and follows immediately from [6, 11].

There is some information about the module 1_X^S where X is the pointwise stabilizer of a finite subset of Ω. Camina and Evans [3] enumerate the submodules of these modules. In particular, we have

THEOREM 3 ([3, Theorem 2.4])): *Let L_k be the permutation PS-module afforded by the action of S on the set of all k point sets of Ω. Then the submodule lattice of L_k is a finite chain.*

Much earlier Latyshev [9, Theorem 2] observed that submodules of L_k satisfy the ascending chain conditon.

3. Razmyslov's ideals in the group rings of symmetric groups

The results of Sections 3 and 4 are due to Razmyslov [12]. The origin of this exposition is Amitsur's lectures [1]; we give more details and stress the group rings of symmetric groups rather than applications to polynomial identities. An additional reason to expose the results of [12] here is that Razmyslov's book is too difficult for the reader who has no special interest in identities of algebras.

For $\sigma \in S_n$ we put

$c_n(s) = -1 + \{\text{the numbers of orbits of the cyclic group } \langle s\rangle \text{ on } \{1,2,\ldots,n\}\}.$

In particular, for $s = e$, the identity permutation, we have $c_n(e) = n-1$.

Let $T = P[t]$ be the polynomial ring in one indeterminate t over a field P.

Definition 1: Let θ: $PS_n \to T$ be the linear function defined on S_n by $\theta(s) = t$ $(s \in S_n)$.

Thus, if $\alpha = \sum_{s\in S_n} a_s s \in PS_n$, then $\theta(\alpha) = \sum_{s\in S_n} a_s t^{c_n(s)}$. Observe that θ is constant on conjugacy classes of PS_n; hence θ can be considered as a function on $Z = Z(PS_n)$, the center of PS_n. Observe that if χ_n is the complex character of the representation of S_n in V_{nn} (see Section 2 above) then $\chi_n(s) = n^{c_n(s)+1}$; see James and Kerber [8, 4.3.4].

Definition 2: Let I be an ideal of $T = P[t]$. We define the Razmyslov ideal $L_n(I)$ as follows:

$$L_n(I) = \left\{ \alpha = \sum_{s\in S_n} a_s s \in PS_n\colon \theta(\alpha g) \in I \text{ for all } g \in S_n \right\}.$$

We shall consider the group ring TS_n of S_n over the polynomial ring $T = P[t]$. We put $d_n = \sum_{s\in S_n} t^{c_n(s)} s \in TS_n$. Observe that d_n is central in TS_n.

PROPOSITION 3: *Let I be an ideal of R and IS_n the corresponding ideal of RS_n. Let $\alpha = \sum_{s\in S_n} a_s s \in PS_n$. The following conditions are equivalent:*

(i) $\alpha \in L_n(I)$;

(ii) $d_n\alpha \in IS_n$.

Proof: We have

$$d_n\alpha = \sum_{r,s\in S_n} a_s t^{c_n(r)} rs = \sum_{q\in S_n} q \left(\sum_{s\in S_n} a_s t^{c_n(qs^{-1})} \right).$$

As $c_n(s) = c_n(s^{-1})$ for $s \in S_n$, the expression in the brackets is equal to $\sum_{s\in S_n} a_s t^{c_n(sq^{-1})}$. If $\alpha \in L_n(I)$ then $\sum_{s\in S_n} a_s t^{c_n(sq^{-1})} \in I$ for all $q \in S_n$ so $d_n\alpha \in IS_n$. Conversely, if $d_n\alpha \in IS_n$, then the coefficient of q in the expression above belongs to I for all q; this means that $\alpha \in L_n(I)$. ∎

Observe that $L_n(I)$ is a two-sided ideal of S_n, as d_n is a central element of RS_n.

The crucial fact is that the correspondence $I \to L_n(I)$ is compatible with the embedding $S_n \to S_{n+1}$.

PROPOSITION 4: *Let $g \in PS_n \subset PS_{n+1}$ (the natural embedding). Then $g \in L_n(I)$ if and only if $g \in L_{n+1}(I)$.*

Prior to proving this, we express d_{n+1} in terms of d_n. To be precise, we shall denote the image of an element x of S_n in S_{n+1} or of an element x of PS_n in PS_{n+1} by $\hat{x}$.

LEMMA 1: *Let δ_i denote the transposition $(i, n+1)$, and e the identity element of a group. Then $d_{n+1} = \hat{d}_n(te + \delta_1 + \cdots + \delta_n) = (te + \delta_1 + \cdots + \delta_n)\hat{d}_n$.*

Proof: We have

$$d_{n+1} = \sum_{s\in S_{n+1}} t^{c_{n+1}(s)} s = \sum_{s\in S_n} t^{c_{n+1}(\hat{s})} \hat{s} + \sum_{i=1}^{n} \sum_{s\in S_n} t^{c_n(\hat{s}\delta_i)} \hat{s}\delta_i.$$

As $c_{n+1}(\hat{s}) = c_n(s) + 1$ and $c_{n+1}(\hat{s}\delta_i) = c_n(s)$ for all i, this equals

$$t \left(\sum_{s\in S_n} t^{c_n(s)} \hat{s} \right) + \sum_{i=1}^{n} \left(\sum_{s\in S_n} t^{c_n(s)} \hat{s} \right) \delta_i$$

$$= \left(\sum_{s\in S_n} t^{c_n(s)} \hat{s} \right) (te + \delta_1 + \cdots + \delta_n) = \hat{d}_n(te + \delta_1 + \cdots + \delta_n).$$

By using left cosets instead of right ones we similarly get the second equality of the Lemma.

Proof of Proposition 4: Obviously, $gd_n \in IS_n$ is equivalent to $\hat{g}\hat{d} \in IS_{n+1}$. Therefore, by Lemma 1, $g \in L_n(I)$ is equivalent to $\hat{g} \in L_{n+1}(I)$. Observe further that $e, \delta_1, \ldots, \delta_n$ form a basis of the free RS_n-module RS_{n+1}; hence it follows from $\hat{g}\hat{d}_n(te + \delta_1 + \cdots + \delta_n) \in IS_{n+1}$ that $\hat{g}\hat{d}_n \in IS_{n+1}$. ∎

COROLLARY 1: $\hat{L}_n(I) \subset L_{n+1}(I)$.

Let us consider PS_n as a subalgebra of PS_{n+1} under the natural embedding. Put $L(I) = \bigcup_n L_n(I)$. It is clear that $L(I)$ is an ideal of PS. For the case $\mathrm{char}(P) = 0$ the ideals $L(I)$ are understood fairly well. For $\mathrm{char}(P) > 0$ only a little is known about these ideals. Razmyslov has obtained some essential information for $I = (t-k)T$ with $1 \le k < p$, and for $I = (t-1)(t-2)\ldots(t-p+1)T$; see [12, §33.2].

PROPOSITION 5: *Let J be a prime ideal of T other than tT. Then the ideal $L(J)$ is prime.*

Proof: Put $I = L(J)$. Let A, B be ideals with $AB \subseteq I$. Let $a \in A$, $b \in B$ be elements such that $a \notin I$, $b \notin I$. Then $a, b \in PS_n$ for suitable n. Since $a \notin I$, $b \notin I$, it follows that $\theta(a\sigma) \notin J$ and $\theta(b\tau) \notin J$ for some $\sigma, \tau \in S_n$. By replacing a and b by $a\sigma$ and $b\tau$, respectively, we can assume that $\theta(a) \notin J$, $\theta(b) \notin J$. Viewing S_n as a subgroup of S_{2n} we can replace b by sbs^{-1}, where $s \in S$ is such that sbs^{-1} fixes the symbols $\{1, \ldots, n\}$. Of course, $\theta(sbs^{-1}) = \theta(b)$ so $sbs^{-1} \notin J$. Let $a = \sum \alpha_k a_k$ and $b = \sum_k \beta_k b_k$, with $\alpha_k, \beta_k \in P$, $a_k, b_k \in S_n$. We have $\theta(ab) \in J$ and $\theta(ab) = \sum_{k,l} \alpha_k \beta_l t^{c_{2n}(a_k b_l)}$. As a fixes the symbols $\{n+1, \ldots, 2n\}$, and b fixes the symbols $\{1, \ldots, n\}$, we have $c_{2n}(a_k b_l) = c_n(a_k) + c_n(b_l) + 1$ for all k, l. It follows that

$$\theta(ab) = \sum_{k,l} \alpha_k \beta_l t^{c_{2n}(a_k b_l)} = \sum_{k,l} \alpha_k \beta_l t^{c_n(a_k) + c_n(b_l) + 1}$$

$$= \left(\sum_k \alpha_k t^{c_n(a_k)}\right)\left(\sum_1 \beta_1 t^{c_n(b_l)}\right) t = \theta(a)\theta(b)t \in J.$$

As J is a prime ideal and $t \notin J$, either $\theta(a)$ or $\theta(b) \in J$. So the Proposition follows. ∎

4. Razmyslov's ideals in the case of characteristic 0

It is well-known that the field of rational numbers $\mathbb{Q}$ is a splitting field for representations of symmetric groups. It follows that central idempotents can be expressed in terms of class sums with rational coefficients. We recall that irreducible representations of S_n over $\mathbb{C}$ or $\mathbb{Q}$ are parametrized by Young diagrams. Let Λ_n denote the set of Young diagrams of size n.

In this section, P is a field of characteristic 0. Let τ_i be the sum in PS_n of all the elements $s \in S_n$ with $c_n(s) = i$. Then $d_n = t^{n-1}e + t^{n-2}\tau_{n-2} + \cdots + \tau_0$. Let $1 = \sum_{\lambda\in\Lambda} e_\lambda$ be the decomposition of the unit element of PS_n. Let us express τ_j in terms of e_λ and collect the coefficients of e_λ. We get

$$d_n = t^{n-1}\cdot 1 + t^{n-2}\tau_{n-2} + \cdots + \tau_0 = \sum_{\lambda\in\Lambda} D_\lambda(t)e_\lambda. \tag{1}$$

It is clear that $D_\lambda(t)$ are polynomials in t with rational coefficients. Let χ_λ be the character of the irreducible representation corresponding to e_λ. Then $\chi_\lambda(e_\mu) = 0$ for $\lambda \neq \mu$ and $\chi_\lambda(e_\lambda) = \chi_\lambda(1)$. Hence we have

$$D_\lambda(t) = \sum_i t^i\chi_\lambda(\tau_i)/\chi_\lambda(1). \tag{2}$$

By multiplying (1) by e_λ we get

$$d_n e_\lambda = D_\lambda(t)e_\lambda. \tag{3}$$

Proposition 6: *Let J be an ideal of $T = P[T]$. Then $PS_ne_\lambda \in L_n(J)$ if and only if $D_\lambda(t) \in J$.*

Proof: If $D_\lambda(t) \in J$ then by (3) $d_ne_\lambda \in JS_n$; by Proposition 3 we have $e_\lambda \in L_n(J)$. Conversely, if $e_\lambda \in L_n(J)$ then let $e_\lambda = \sum_{s\in S_n} a_s s$ with $a_s \in P$; then $d_ne_\lambda = \sum_s D_\lambda(t)a_s s \in JS_n$, which implies that $D_\lambda(t) \in J$. ∎

Corollary 2: *Let $\lambda \subseteq \mu$ be diagrams. Then $D_\mu(t)$ is a multiple of $D_\lambda(t)$.*

Proof: Take $J = D_\lambda(t)T$. Let $n = |\lambda|$, $m = |\mu|$. By (3), $d_ne_\lambda = D_\lambda(t)e_\lambda \subset JS_n$, and by Proposition 3, $e_\lambda \in L_n(J)$; so $PS_me_\lambda PS_m \subseteq L_m(J)$ as $L(J)$ is a two-sided ideal. It is well known that $\lambda \subseteq \mu$ implies that $e_\mu \in PS_me_\lambda PS_m$. Hence $PS_me_\mu \subseteq PS_me_\lambda PS_m \subseteq L_m(J)$. By Proposition 3, $D_\mu(t) \in J$, so $D_\mu(t)$ is a multiple of $D_\lambda(t)$. ∎

In fact, $D_\mu(t)$ is a multiple of $D_\lambda(t)$ if and only if $\lambda \subseteq \mu$. However, we are not yet ready to prove this (see Corollary 5 below).

COROLLARY 3: *$L_n(J)$ is the direct sum of those ideals PS_ne_λ such that $D_\lambda(t) \in J$.*

Let λ be a diagram. Let us fill λ with integers so that we place the number $i-j$ into the box located in the i-th row and the j-th column. To λ we may associate the polynomial $C_\lambda(t)$, called the content of λ, whose roots are just the numbers in the boxes of λ defined above. So, if t is an indeterminate, then $C_\lambda(t) = \prod(t-i+j)$ with the product taken over all boxes of λ. The notion of the diagram content was probably introduced by Robinson and Thrall [14]; see also [13, §4.3]. It is clear that λ is completely determined by its content (see [13, 4.33]).

THEOREM 4 ([12, Lemma 26.6]): $tD_\lambda(t) = C_\lambda(t)$.

COROLLARY 4: *$D_\lambda(t)$ determines λ (that is, $D_\lambda(t)$ are different for different λ's).*

Observe that Corollary 4 follows, provided the collection

$$\{\chi_\lambda(\tau_i)/\chi_\lambda(1)\}_{0\le i\le n-1}$$

determines λ (see (2)). This fact is contained in Benson and Curtis [2, Theorem 6.7]. It follows also from a result of Farahat and Higman [5] that the class sums are polynomials with integer coefficients in τ_i's. Indeed, as the τ_i's are scalars in the representation afforded by χ_λ, we have $\chi_\lambda\left(\prod_i \tau_i^{r_i}\right)/\chi_\lambda(1) = \prod_i(\chi_\lambda(\tau_i)/\chi_\lambda(1))^{r_i}$. It follows that $\{\chi_\lambda(\tau_i)/\chi_\lambda(1)\}$ determines the "normalized" character $\chi_\lambda/\chi_\lambda(1)$. In turn, the latter determines χ_λ, as $(\chi_\lambda/\chi_\lambda(1), \chi_\lambda/\chi_\lambda(1)) = 1/(\chi_\lambda(1))^2$.

COROLLARY 5: *If $D_\mu(t)$ is a multiple of $D_\lambda(t)$, then $\lambda \subseteq \mu$.*

Proof of Theorem 4: By induction on $n = |\lambda|$. For $n=1$ we have $D_\lambda(t) = 1$. Suppose that the Theorem is true for all proper subdiagrams of λ. Let us consider first the case where λ is not rectangular.

$$\begin{matrix} 0 & -1 & -2 & \dots & -s & -s-1 \\ 1 & 0 & -1 & \dots & \dots & \\ \dots & \dots & \dots & \dots & & \\ r & r-1 & \dots & & & \end{matrix}$$

Then λ has at least two corner boxes, say, (k', l') and (k'', l''). Let λ' and λ'' be the diagrams obtained from λ by omitting the box (k', l') and (k'', l''), respectively. By the induction assumption $tD_{\lambda'}(t) = C_{\lambda'}(t)$ and $tD_{\lambda''}(t) = C_{\lambda''}(t)$. By (2),

$D_\lambda(t)$ is of degree n, and by Corollary 2, $tD_\lambda(t)$ is a multiple of $tD_{\lambda'}(t) = C_{\lambda'}(t)$ and of $tD_{\lambda''}(t) = C_{\lambda''}(t)$. It follows that $tD_\lambda(t) = C_{\lambda'}(t)(t-\alpha)$ and $tD_\lambda(t) = C_{\lambda''}(t)(t-\beta)$ where $\alpha, \beta \in \mathbb{C}$. It follows that α is the root of the polynomial $C_{\lambda''}(t)$ which is not a root of $C_{\lambda'}(t)$ while β is the root of $C_{\lambda'}(t)$ which is not a root of $C_{\lambda''}(t)$. This implies the desired assertion when λ is not rectangular.

The case where λ is rectangular is more difficult. Let (k,l) be its unique corner box. Let λ' be the subdiagram of λ, obtained by omitting the box (k,l) from λ. By the induction assumption, $D_\lambda(t) = C_{\lambda'}(t)(t-\alpha)$ with $\alpha \in \mathbb{C}$; however, what is the value of α?

In order to answer, we calculate the sum σ of roots of $D_\lambda(t)$ in two ways. On the one hand, $\sigma = -l\binom{k}{2} + k\binom{l}{2} - (k-l) + \alpha$, as the sum of roots of $C_{\lambda'}(t)$ is just $-l\binom{k}{2} + k\binom{l}{2}$. On the other hand, $-\sigma$ is equal to the coefficient of t^{kl-2} of $D_\lambda(t)$. By formula (2), this coefficient is $\chi_\lambda(\tau)/\chi_\lambda(1)$, where τ is the sum of all the elements $\sigma \in KS_n$ such that $c_n(\sigma) = n-1$. These are just the transpositions, so τ_{kl-2} is the sum of all transpositions. Since the number of transpositions of S_{kl} is $\binom{kl}{2}$, then $\chi_\lambda(\tau_{kl-2}) = \chi_\lambda(\zeta)\binom{kl}{2}$ where ζ is a transposition. According to Murnaghan [10, ch.5, §1] $\chi_\lambda(\zeta)/\chi_\lambda(1) = (l-k)/(kl-1)$ so $\chi_\lambda(\tau_{kl-2})/\chi_\lambda(1) = kl(l-k)/2 = -l\binom{k}{2} + k\binom{l}{2}$. Comparing both the values for σ, we get $\alpha = k-l$. This completes the proof. ∎

Now we can clarify which are the ideals $L(I)$. Observe that each ideal of T is generated by a single element. Let $I = E(t)T$ where $E(t)$ is a polynomial. If $L(I) \neq \{0\}$ then we have $e_\lambda \in L(I)$ for some diagram λ. By Proposition 6, $D_\lambda(t) \in I$. So $D_\lambda(t)$ is a multiple of $E(t)$. In particular, the roots of $E(t)$ are integers. Let us locate them into the tableau by placing the root $i \in \mathbb{Z}$ of multiplicity m_i into the i-th diagonal with no gaps (however, the root $i=0$ has to be inserted m_0+1 times). After this we find the least Young diagram λ which contains all the numbers inserted.

This shows that there exists only one such diagram. This implies

THEOREM 5: *Let I be a proper nonzero ideal of T. Then $L(I) = L(D_\lambda(t)T)$ for λ obtained as above.*

The important question is for which ideals J of T is the Razmyslov ideal $L(J)$ nonzero. Obviously, it suffices to solve it for prime J. The answer is available in case P is of characteristic zero.

COROLLARY 6: *$L(J) \neq \{0\}$ if and only if J is generated by a polynomial with integer roots.*

I do not know whether this asserion is true for prime characteristic.

THEOREM 6: *Suppose P is of characteristic 0, and J a proper nonzero ideal of T. Then $L(J)$ is prime if and only if $J = D_\lambda(t)T$ where λ is a rectangular diagram.*

Proof: Suppose $L(J)$ is a prime ideal. By Theorem 5, $L(J) = L(D_\lambda(t)T)$ for some diagram λ. Suppose that λ is not rectangular. Then λ is a union of two proper diagrams, say, λ' and λ''. It is clear that both $L(D_{\lambda'}(t)T)$ and $L(D_{\lambda''}(t)T)$ contain and do not coincide with $L(J)$. The product of these ideals is contained in their intersection. So it suffices to show that the intersection is contained in $L(J)$. If $x \in PG$ is in the intersection then $\theta(x)$ lies both in $D_{\lambda'}(t)$ and $D_{\lambda''}(t)$. As $D_\nu(t) = C_\nu(t)/t$ for any diagram ν, it follows that $\theta(x)$ is a multiple of $D_\lambda(t)$. Hence $x \in L(J)$. So $L(D_{\lambda'}(t)T) \cdot L(D_{\lambda''}(t)T) \subseteq L(J)$, a contradiction. ∎

It would be very interesting to answer the question for which J the ideal $L(J)$ is prime, when P has prime characteristic. In view of Theorem 6 one cannot expect this to be the case only if J is prime.

Addendum

The notion of the content of a diagram plays an important part in the theory of p-blocks of p-modular representations of S_n. In particular, we have:

THEOREM 7 (see Robinson [13, 5.36 and 5.42])): *Let ϕ_λ, ϕ_μ be ordinary irreducible representations of S_n afforded by the diagrams λ, μ, respectively. Then ϕ_λ, ϕ_μ belong to the same p-block if and only if $C_\lambda(t) \equiv C_\mu(t) \pmod p$.*

We can give a new proof of this theorem with the aid of a theorem of Farahat and Higman [5] and the following formula (see (2) and Theorem 4):

$$C_\lambda(t) = tD_\lambda(t) = \sum_{i=0}^{n-1} t^{i+1}\chi_\lambda(\tau_i)/\chi_\lambda(1).$$

Let ξ_λ denote the class sum afforded by the conjugacy class labeled by the diagram λ. Farahat–Higman's theorem says that ξ_λ's are expressed in terms of τ_i as polynomials with integer nonnegative coefficients. Recall that irreducible representations ϕ_λ, ϕ_μ of S_n belong to the same p-block if and only if

$\chi_\lambda(\xi_\nu)/\chi_\lambda(1) \equiv \chi_\mu(\xi_\nu)/\chi_\mu(1) \pmod p$ for each ν (see, for example, [4, 85.12 and 85.13]).

Proof of Theorem 7: It follows from formula (2) of Section 4 that $D_\lambda(t) \equiv D_\mu(t) \pmod p$ if $\chi_\lambda(\xi_\nu)/\chi_\lambda(1) \equiv \chi_\mu(\xi_\nu)/\chi_\mu(1) \pmod p$, as the τ_i's are sums of ξ_ν's with integer coefficients. Conversely, if $D_\lambda(t) \equiv D_\mu(t) \pmod p$, then $\chi_\lambda(\tau_i)/\chi_\lambda(1) \equiv \chi_\mu(\tau_i)/\chi_\mu(1) \pmod p$. As the ξ_ν's are polynomials in τ_i with integer coefficients, it suffices to show that the $\chi_\lambda(\xi_\nu)/\chi_\lambda(1)$ are polynomials in $\chi_\lambda(\tau_i)/\chi_\lambda(1)$. Observe that the $\phi_\lambda(\tau_i)$ are scalar matrices, by Schur's lemma. It follows that $\chi_\lambda\left(\prod_i \tau_i^{r_i}\right)/\chi_\lambda(1) = \prod_i(\chi_\lambda(\tau_i)\chi_\lambda(1))^{r_i}$. Let $\xi_\nu = \sum a_{i\nu}\left(\prod_i \tau_i^{r_{i\nu}}\right)$ with integer a_{ij}. Then

$$\chi_\lambda(\xi_\nu(1) = \sum a_{i\nu}\chi_\lambda\left(\prod_i \tau_i^{r_{i\nu}}\right)/\chi_\lambda(1) = \sum a_{i\nu}(\chi_\lambda(\tau_i)/\chi_\lambda(1))^{r_{i\nu}},$$

as desired.

Additional Note: After the paper was submitted, there appeared some promising ideas and results which deserve to be mentioned here. It was observed in Theorem 3.5 [A. E. Zalesskii, *Group rings of simple locally finite groups*, NATO ASI Series C, Vol. 471, Kluwer, Dordrecht–Boston–London, 1995, pp. 219–246] that semiprimitive ideals of PS can be described in terms of representations of S_n, $n = 5, 6, \ldots$. To use this approach, one needs information about the branching rule for modular representations of S_n. A weaker form of the branching rule was recently obtained by Kleshchev (see [A. S. Kleshchev, *Branching rule for modular representations of symmetric groups, II*, Journal für die reine und angewandte Mathematik **459** (1995), 163–212, Theorem 0.5]). By using this result Kleshchev constructed new examples of maximal ideals of PS [A. S. Kleshchev, *Completely splittable representations of symmetric groups*, Journal of Algebra (to appear)].

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